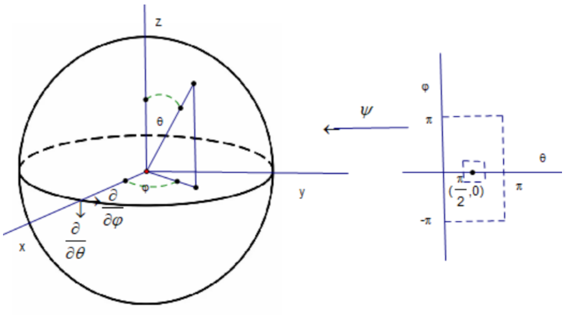
Killing vector fields

§ 02 S^2 上的 Killing vector field



$\psi:(0,\pi)\times(-\pi,\pi)\to S^2$ given by $\psi(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$

Parameterizes a neighorhood of the point $(1,0,0)=\psi(\frac{\pi}{2},0)$

我們注意到$\psi(\theta,0)\to$通過(1,0,0)的經線，$\psi(0,\varphi)\to$赤道。

$$ds^2=d\theta^2+\sin^2\theta d\varphi^2$$

$$x=\sin\theta\cos\varphi, y=\sin\theta\sin\varphi, z=\cos\theta$$

$$\frac{\partial}{\partial\varphi}=\frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y}+\frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}$$

$$=-\sin\theta\sin\varphi\frac{\partial}{\partial x}+\sin\theta\cos\varphi\frac{\partial}{\partial y}+0=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

$R=\frac{\partial}{\partial\varphi}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$ is a vector field which generates rotation about the z-axis，is an isometry and a Killing vector field that preserves the metric，i.e. $L_X g=0$。

$S=\cos\varphi\partial_\theta-\cot\theta\sin\varphi\partial_\varphi$

$T=-\sin\varphi\partial_\theta-\cot\theta\cos\varphi\partial_\varphi$ are Killing vectors。

The flow of $\frac{\partial}{\partial\varphi}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$ is $\varphi_t=(x\cos t-y\sin t, x\sin t+y\cos t, z)$，and the matrix of the rotation about the z-axis is $\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}$，$\{\varphi_t | t\in R\}\cong SO(2,R)$

$$\frac{d}{d\theta}\Big|_{\theta=0}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-y\partial_x+x\partial_y$$

But $\frac{\partial}{\partial\theta}=\cos\theta\cos\varphi\frac{\partial}{\partial x}+\cos\theta\sin\varphi\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial z}$ is not a Killing vector field。

$(\frac{\partial}{\partial\theta})_{(1,0,0)}\leftrightarrow(0,0,-1)$朝下的向量

$$X=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}=\frac{\partial}{\partial\varphi}\qquad Y=\frac{\partial}{\partial\theta}=\cos\theta\cos\varphi\frac{\partial}{\partial x}+\cos\theta\sin\varphi\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial z}$$

$$g=dx^2+dy^2+dz^2=d\theta^2+\sin^2\theta d\varphi^2$$

$$(L_X g)_{\mu\nu}=X^\rho\partial_\rho g_{\mu\nu}+\partial_\mu X^\rho g_{\rho\nu}+\partial_\nu X^\rho g_{\rho\mu}$$

經過小心計算結果

$(L_X g)_{\mu\nu}=0,(L_Y g)_{22}=2\sin\theta\cos\theta$，所以 X 是 Killing field，Y 不是。